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Bethe ansätze for 19-vertex models

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Abstract. The 19-vertex models of Zalomodchikov–Fateev, Izergin–Korepin and the supersymmetric $osp(1|2)$ with periodic boundary conditions are studied. We find the spectrum of these quantum spin chains using the coordinate Bethe ansatz. The approach is a suitable parametrization of their wavefunctions. We also applied the algebraic Bethe ansatz in order to obtain the eigenvalues and eigenvectors of the corresponding transfer matrices.

1. Introduction

One-dimensional quantum spin-chain Hamiltonians and classical statistical systems in two spatial dimensions on a lattice (vertex models), share a common mathematical structure responsible for our understanding of these integrable models [1, 2]. If the Boltzmann weights underlying the vertex models are obtained from solutions of the Yang–Baxter (YB) equation the commutativity of the associated transfer matrices follows immediately, leading to their integrability.

The Bethe ansatz (BA) is the powerful method in the analysis of integrable quantum models. There are several versions: coordinate BA [3], algebraic BA [4], analytical BA [5], etc developed for diagonalization of the corresponding Hamiltonian.

The simplest version is the coordinate BA. In this framework one can obtain the eigenfunctions and the spectrum of the Hamiltonian from its eigenvalue problem. It is really simple and clear for the two-state models like the six-vertex models but becomes awkward for models with a higher number of states.

The algebraic BA, also known as as quantum inverse scattering method, is an elegant and important generalization of the coordinate BA. It is based on the idea of constructing eigenfunctions of the Hamiltonian via creation and annihilation operators acting on a reference state. Here one uses the miraculous fact that the YB equation can be recast in the form of commutation relations for the matrix elements of the monodromy matrix, which play the role of creation and annihilation operators. From this monodromy matrix we obtain the transfer matrix which, by construction, commutes with the Hamiltonian. Thus, constructing eigenfunctions of the transfer matrix determines the eigenfunctions of the Hamiltonian.

On imposing appropriate boundary conditions the BA method leads to a system of equations, the BA equations, which are useful in the thermodynamic limit. The energy of the ground state and its excitations, velocity of sound, etc, may be calculated in this limit. Moreover, in recent years we have witnessed another very fruitful connection between the BA method and conformal field theory. Using the algebraic BA, Korepin [6] found various representations of correlators in integrable models and more recently Babujian and Flume [7]

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developed a method from the algebraic BA which reveals a link to the Gaudin model and renders in the quasiclassical limit, solutions of the Knizhnik–Zamolodchikov equations for the $SU(2)$ Wess–Zumino–Novikov–Witten conformal theory.

Integrable quantum systems containing Fermi fields have been attracting increasing interest due to their potential applications in condensed matter physics. The prototypical examples of such systems are the supersymmetric generalizations of the Hubbard and t–J models [8]. They lead to a generalization of the YB equation associated with the introduction of a Z_2 grading [9], which leads to the appearance of additional signs in the YB equation.

In this paper we consider the coordinate and algebraic versions of the BA for the trigonometric three-state vertex models of 19 vertices with periodic boundary conditions. These models are well known in the literature: the Zamolodchikov–Fateev (ZF) model or A_1^1 model [10], the Izergin–Korepin (IK) model or A_2^2 model [11] and the supersymmetric $osp(1|2)$ model [12].

While the BA solution of the periodic ZF model was derived by a fusion procedure [13] in [14, 15], a generalization of the algebraic BA was developed by Tarasov [16] to solve the IK model. Moreover, the IK model was solved via coordinate BA by Batchelor *et al* in [17].

In the context of the algebraic BA, the version presented here is based on the Tarosov approach but now we include the ZF model and we also extend it to the graded version of the quantum inverse scattering method in order to consider the $osp(1|2)$ model.

In the context of the coordinate BA, we propose here a new parametrization of wavefunctions. This result is of fundamental importance since it allows us to treat these 19-vertex models in the same way and the coordinate BA for these three-states models becomes simple enough as for two-state models.

The main goal in this paper is to reveal the common structure of these 19-vertex models which permits us to apply the BA method, unifying old and new results.

The paper is organized as follows. In section 2 we present the models. In section 3 the spectra of the corresponding Hamiltonians are derived using the coordinate BA and in section 4 the algebraic BA is also used to diagonalize the corresponding transfer matrices. We justify this twofold presentation remarking that the BA method is apparently version dependent. It means that when one solves a model using a particular BA version is not always clear how to extend the solution for all possible versions. For example, the biquadratic model was solved by coordinate BA in [18, 19] and its algebraic BA version is still unknown. Finally, the conclusions are reserved for section 5.

2. Description of the models

Let us start with the graded formulation and then recover the non-graded formulation from it.

Consider $V = V_0 \oplus V_1$ a Z_2-graded vector space where 0 and 1 denote the even and odd parts, respectively. The multiplication rules in the graded tensor product space $V \overset{s}{\otimes} V$ differ from the ordinary ones by the appearance of additional signs. The components of a linear operator $A \overset{s}{\otimes} B \in V \overset{s}{\otimes} V$ result in matrix elements of the form

$$(A \overset{s}{\otimes} B)^{\gamma\delta}_{\alpha\beta} = (-)^{p(\beta)(p(\alpha)+p(\gamma))}\, A_{\alpha\gamma} B_{\beta\delta}. \tag{2.1}$$

The action of the graded permutation operator $\mathcal{P}$ on the vector $|\alpha\rangle \overset{s}{\otimes} |\beta\rangle \in V \overset{s}{\otimes} V$ is defined by

$$\mathcal{P}\,|\alpha\rangle \overset{s}{\otimes} |\beta\rangle = (-)^{p(\alpha)p(\beta)} |\beta\rangle \overset{s}{\otimes} |\alpha\rangle \quad \Longrightarrow \quad (\mathcal{P})^{\gamma\delta}_{\alpha\beta} = (-)^{p(\alpha)p(\beta)} \delta_{\alpha\delta}\, \delta_{\beta\gamma} \tag{2.2}$$

where $p(\alpha) = 1$ (0) if $|\alpha\rangle$ is an odd (even) element.

The central object in the theory of integrable models is the $\mathcal{R}$-matrix $\mathcal{R}(\lambda)$, where λ is the spectral parameter. It acts on the tensor product $V^1 \otimes V^2$ for a given vector space V and it is a solution of the Yang–Baxter (YB) equation

$$\mathcal{R}_{12}(\lambda)\,\mathcal{R}_{13}(\lambda+\mu)\,\mathcal{R}_{23}(\mu) = \mathcal{R}_{23}(\mu)\,\mathcal{R}_{13}(\lambda+\mu)\,\mathcal{R}_{12}(\lambda) \tag{2.3}$$

in $V^1 \otimes V^2 \otimes V^3$, where $\mathcal{R}_{12} = \mathcal{R} \otimes \mathcal{I}$, $\mathcal{R}_{23} = \mathcal{I} \otimes \mathcal{R}$, etc.

In the graded case, $\mathcal{R}_{13}$, however, does not act trivially on the second space due to signs generated by commuting odd operators. In this case, the graded YB equation in components reads

$$\begin{aligned}&\mathcal{R}^{kk'}_{ii'}(\lambda)\,\mathcal{R}^{jk''}_{ki''}(\lambda+\mu)\,\mathcal{R}^{j'j''}_{k'k''}(\mu)(-)^{p(k')(p(i'')+p(k''))}\\ &\qquad = \mathcal{R}^{k'k''}_{i'i''}(\mu)\,\mathcal{R}^{kj''}_{ik''}(\lambda+\mu)\,\mathcal{R}^{jj'}_{kk'}(\lambda)(-)^{p(k')(p(j'')+p(k''))}.\end{aligned} \tag{2.4}$$

Besides $\mathcal{R}$ we have to consider matrices $R = \mathcal{P}R$ which satisfy

$$R_{12}(\lambda)\,R_{23}(\lambda+\mu)\,R_{12}(\mu) = R_{23}(\mu)\,R_{12}(\lambda+\mu)\,R_{23}(\lambda). \tag{2.5}$$

Because only R_{12} and R_{23} are involved, equation (2.5) written in components looks the same as in the non-graded case. Moreover, the matrices $\mathcal{R}_{ng} = PR$ satisfy the ordinary YB equation (2.3) where P is the non-graded permutation operator.

2.1. The R-matrices

We will consider 19-vertex models for which their R-matrices have a common form

$$R(\lambda) = \begin{pmatrix}
x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & y_5 & 0 & x_2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & y_7 & 0 & y_6 & 0 & x_3 & 0 & 0\\
0 & x_2 & 0 & x_5 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \varepsilon y_6 & 0 & \varepsilon x_4 & 0 & \varepsilon x_6 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & y_5 & 0 & x_2 & 0\\
0 & 0 & x_3 & 0 & x_6 & 0 & x_7 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & x_2 & 0 & x_5 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1
\end{pmatrix}. \tag{2.6}$$

Here we have assumed that the grading of threefold space is $p(1) = 1$, $p(2) = \varepsilon$ and $p(3) = 1$, where $\varepsilon = \pm$. The matrix elements x_i and y_i for each model will be listed below.

ZF R-matrix. The simplest 19-vertex model is the ZF model or A_1^1 model. The solution of the YB equation was found in [10]. It can also be constructed from the six-vertex model using the fusion procedure. The corresponding R-matrix has the form (2.6) with $\varepsilon = 1$ and

$$\begin{aligned}
x_1(\lambda) &= \sinh(\lambda+\eta)\sinh(\lambda+2\eta)\\
x_2(\lambda) &= \sinh\lambda\sinh(\lambda+\eta)\\
x_3(\lambda) &= \sinh\lambda\sinh(\lambda-\eta)\\
x_5(\lambda) &= y_5(\lambda) = \sinh(\lambda+\eta)\sinh 2\eta\\
x_6(\lambda) &= y_6(\lambda) = \sinh\lambda\sinh 2\eta\\
x_7(\lambda) &= y_7(\lambda) = \sinh\eta\sinh 2\eta\\
x_4(\lambda) &= x_2(\lambda) + x_7(\lambda).
\end{aligned} \tag{2.7}$$

IK R-matrix. The solution of this YB equation was found in [11]. It cannot be constructed from the six-vertex model using the fusion procedure. The R-matrix has the form (2.6) with $\varepsilon = 1$ and

$$\begin{aligned}
x_1(\lambda) &= \sinh(\lambda - 5\eta) + \sinh\eta \\
x_2(\lambda) &= \sinh(\lambda - 3\eta) + \sinh 3\eta \\
x_3(\lambda) &= \sinh(\lambda - \eta) + \sinh\eta \\
x_4(\lambda) &= \sinh(\lambda - 3\eta) + \sinh 3\eta - \sinh 5\eta + \sinh\eta \\
x_5(\lambda) &= -\sinh 2\eta\left(\mathrm{e}^{-\lambda+3\eta} + \mathrm{e}^{-3\eta}\right) \\
y_5(\lambda) &= -\sinh 2\eta\left(\mathrm{e}^{\lambda-3\eta} + \mathrm{e}^{3\eta}\right) \\
x_6(\lambda) &= \mathrm{e}^{2\eta}\sinh 2\eta\left(1 - \mathrm{e}^{-\lambda}\right) \\
y_6(\lambda) &= \mathrm{e}^{-2\eta}\sinh 2\eta\left(1 - \mathrm{e}^{\lambda}\right) \\
x_7(\lambda) &= -2\mathrm{e}^{-\lambda+2\eta}\sinh\eta\sinh 2\eta - \mathrm{e}^{-\eta}\sinh 4\eta \\
y_7(\lambda) &= 2\mathrm{e}^{\lambda-2\eta}\sinh\eta\sinh 2\eta - \mathrm{e}^{\eta}\sinh 4\eta.
\end{aligned} \tag{2.8}$$

$Osp(1|2)$ *R-matrix.* The trigonometric solution of the graded YB equation for the fundamental representation of $osp(1|2)$ algebra was found by Bazhanov and Shadrikov in [12]. It has the form (2.6) with $\varepsilon = -1$ and

$$\begin{aligned}
x_1(\lambda) &= \sinh(\lambda + 2\eta)\sinh(\lambda + 3\eta) \\
x_2(\lambda) &= \sinh\lambda\sinh(\lambda + 3\eta) \\
x_3(\lambda) &= \sinh\lambda\sinh(\lambda + \eta) \\
x_4(\lambda) &= \sinh\lambda\sinh(\lambda + 3\eta) - \sinh 2\eta\sinh 3\eta \\
x_5(\lambda) &= \mathrm{e}^{-\lambda/3}\sinh 2\eta\sinh(\lambda + 3\eta) \\
y_5(\lambda) &= \mathrm{e}^{\lambda/3}\sinh 2\eta\sinh(\lambda + 3\eta) \\
x_6(\lambda) &= -\mathrm{e}^{-\lambda/3-2\eta}\sinh 2\eta\sinh\lambda \\
y_6(\lambda) &= \mathrm{e}^{\lambda/3+2\eta}\sinh 2\eta\sinh\lambda \\
x_7(\lambda) &= \mathrm{e}^{\lambda/3}\sinh 2\eta\left(\sinh(\lambda + 3\eta) + \mathrm{e}^{-\eta}\sinh\lambda\right) \\
y_7(\lambda) &= \mathrm{e}^{-\lambda/3}\sinh 2\eta\left(\sinh(\lambda + 3\eta) + \mathrm{e}^{\eta}\sinh\lambda\right).
\end{aligned} \tag{2.9}$$

The rational limit of (2.9) is well known in the literature [20] and can be written in the form

$$R(\lambda, \eta) = \eta\mathcal{I} + \lambda\mathcal{P} + \frac{\lambda\eta}{\lambda + \frac{3}{2}\eta}\mathcal{U} \tag{2.10}$$

where $\mathcal{I}$ is the identity operator, $\mathcal{P}$ is the graded permutation operator (2.2) and $\mathcal{U}$ is the rank-one projector $\mathcal{U}^2 = \mathcal{U}$. The algebraic solution of (2.10) was obtained by Martins [21], as a limit of the algebraic solution of the IK model.

2.2. The Hamiltonians

In order to derive the Hamiltonian, it is convenient to expand the R-matrix around the regular point $\lambda = 0$. For the 19-vertex models the corresponding solutions with the standard normalization can be read directly from (2.6). They have the form

$$R(\lambda, \eta) \sim \mathcal{I} + \lambda(\alpha^{-1}\mathcal{H} + \beta\mathcal{I}) + \mathrm{o}(\lambda^2) \tag{2.11}$$

where α and β are scalar functions.

The Hamiltonian is a local sum given by

$$H = \sum_{k=1}^{N-1} H_{k,k+1} + H_{N,1} \tag{2.12}$$

where $H_{k,k+1}$ is the $\mathcal{H}$ in (2.11) acting on the quantum spaces at sites k and $k+1$. Using a spin language, this is a spin-1 Hamiltonian. In the basis where S_k^z is diagonal with eigenvectors $|+,k\rangle, |0,k\rangle, |-,k\rangle$ and eigenvalues $1, 0, -1$, respectively, the Hamiltonian densities acting on two neighbouring sites are then given by

$$H_{k,k+1} = \begin{array}{c} \\ |{+}{+}\rangle \\ |{+}0\rangle \\ |{+}{-}\rangle \\ |0{+}\rangle \\ |00\rangle \\ |0{-}\rangle \\ |{-}{+}\rangle \\ |{-}0\rangle \\ |{-}{-}\rangle \end{array} \begin{pmatrix} z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{z}_5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{z}_7 & 0 & \bar{z}_6 & 0 & z_3 & 0 & 0 \\ 0 & 1 & 0 & z_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon\bar{z}_6 & 0 & \varepsilon z_4 & 0 & \varepsilon z_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{z}_5 & 0 & 1 & 0 \\ 0 & 0 & z_3 & 0 & z_6 & 0 & z_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & z_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_1 \end{pmatrix}_{k,k+1} \tag{2.13}$$

where the matrix elements for each model are:

ZF Hamiltonian. For the ZF model the corresponding quantum spin chain is the spin-1 XXZ model. The two-site Hamiltonian is derived from (2.11) and has the form (2.13) with

$$\begin{array}{lll} \varepsilon = 1 & \alpha = \sinh 2\eta & \beta = 0 \\ z_1 = 0 & z_3 = -1 & z_4 = -2\cosh 2\eta \\ z_5 = \bar{z}_5 = -\cosh 2\eta & z_6 = \bar{z}_6 = 2\cosh\eta & z_7 = \bar{z}_7 = -1 - 2\cosh 2\eta. \end{array} \tag{2.14}$$

IK Hamiltonian. In the IK model the two-site Hamiltonian for the corresponding quantum chain has the form (2.13) with

$$\begin{array}{lll} \varepsilon = 1 & \alpha = -2\sinh 2\eta & \beta = 0 \\ z_1 = 0 & z_3 = \dfrac{\cosh\eta}{\cosh 3\eta} & z_4 = -2\dfrac{\sinh 4\eta \sinh\eta}{\cosh 3\eta} \\ z_5 = -\mathrm{e}^{-2\eta} & \bar{z}_5 = -\mathrm{e}^{2\eta} & \\ z_6 = \mathrm{e}^{2\eta}\dfrac{\sinh 2\eta}{\cosh 3\eta} & \bar{z}_6 = -\mathrm{e}^{-2\eta}\dfrac{\sinh 2\eta}{\cosh 3\eta} & \\ z_7 = -\mathrm{e}^{-4\eta}\dfrac{\cosh\eta}{\cosh 3\eta} & \bar{z}_7 = -\mathrm{e}^{4\eta}\dfrac{\cosh\eta}{\cosh 3\eta}. & \end{array} \tag{2.15}$$

$Osp(1|2)$ Hamiltonian. The two-site quantum Hamiltonian associated with the $osp(1|2)$ model has the form (2.13) with

$$\begin{aligned}
&\varepsilon = -1 && \alpha = \sinh 2\eta && \beta = -\coth 2\eta \\
&z_1 = \cosh 2\eta && z_3 = \frac{\sinh \eta}{\sinh 3\eta} && z_4 = 1 + \coth 3\eta \sinh 2\eta \\
&z_5 = -\frac{\sinh 2\eta}{3} && \bar{z}_5 = -z_5 && \\
&z_6 = -\mathrm{e}^{-2\eta}\frac{\sinh 2\eta}{\sinh 3\eta} && \bar{z}_6 = \mathrm{e}^{2\eta}\frac{\sinh 2\eta}{\sinh 3\eta} && \\
&z_7 = \frac{\sinh 2\eta}{3} + \mathrm{e}^{-\eta}\frac{\sinh 2\eta}{\sinh 3\eta} && \bar{z}_7 = -\frac{\sinh 2\eta}{3} + \mathrm{e}^{\eta}\frac{\sinh 2\eta}{\sinh 3\eta}. &&
\end{aligned} \tag{2.16}$$

Having now built a common ground for these models, we may proceed to find their spectra. We begin with the coordinate BA because of its simplicity.

3. The coordinate Bethe ansatz

In this section results are presented for a periodic quantum spin chain of N atoms each with spin 1 described by the Hamiltonian (2.12). At each site, the *spin variable* may be +1, 0, −1, so that the Hilbert space of the spin chain is $\mathcal{H}^{(N)} = \otimes^N V$ where $V = C^3$ with basis $\{|+\rangle, |0\rangle, |-\rangle\}$. The dimension of the Hilbert space is $\dim \mathcal{H}^{(N)} = 3^N$. On $\mathcal{H}^{(N)}$ we consider the Hamiltonians presented in the previous section.

From (2.12) one can see that H commutes with the operator which shifts the states of the chain by one unit. This means translational invariance of H. Moreover, the Hamiltonian (2.12) preserves the third component of the *spin*,

$$\left[H, S_T^z\right] = 0 \qquad S_T^z = \sum_{k=1}^{N} S_k^z. \tag{3.1}$$

This allows us to divide the Hilbert space of states into different sectors, each labelled by the eigenvalue of the operator number $r = N - S_T^z$. We shall denote by $\mathcal{H}_n^{(N)}$ the subspace of $\mathcal{H}^{(N)}$ with $r = n$. One can easily see that $\dim \mathcal{H}^{(N)} = \sum_{r=0}^{N} \dim \mathcal{H}_r^{(N)}$ with

$$\dim \mathcal{H}_r^{(N)} = \sum_{k=0}^{\left[\frac{1}{2}r\right]} \begin{pmatrix} N \\ r-2k \end{pmatrix} \begin{pmatrix} N-r+2k \\ k \end{pmatrix} \tag{3.2}$$

where $\left[\frac{1}{2}r\right]$ means the integer part of $\frac{1}{2}r$.

3.1. Sector $r = 0$

The sector $\mathcal{H}_0^{(N)}$ contains only one state, the *reference state*, with all spin values equal to +1, $\Psi_0 = \prod_k |+, k\rangle$, satisfying $H\Psi_0 = E_0\Psi_0$, with $E_0 = Nz_1$. All other energies will be measured relative to this state. It means that we will seek eigenstates of H satisfying $(H - Nz_1)\,\Psi_r = \epsilon_r \Psi_r$, in every sector r.

3.2. Sector $r = 1$

In $\mathcal{H}_1^{(N)}$, the subspace of states with all spin values equal to +1 except one with value 0. There are N states $|k[0]\rangle = \left|+ + + \underset{k}{0} + + \cdots + \right\rangle$ which span a basis of $\mathcal{H}_1^{(N)}$. The ansatz for the

eigenstate is thus of the form

$$\Psi_1 = \sum_{k=1}^{N} A(k)\,|k[0]\rangle. \tag{3.3}$$

The unknown wavefunction $A(k)$ determines the probability that the *spin variable* has the value 0 at the site k.

From the complete invariance translational due to the periodic boundary conditions, it follows that $A(k)$ is just the wavefunction for a plane wave

$$A(k) = \xi^k \tag{3.4}$$

where $\xi = \mathrm{e}^{\mathrm{i}\theta}$, θ being some particular momentum fixed by the boundary condition $A(N+k) = A(k)$.

When H acts on $|k[0]\rangle$, it sees the reference configuration, except in the vicinity of k, and using (2.13) we obtain the eigenvalue equations

$$(\epsilon_1 + 2z_1 - z_5 - \bar{z}_5)\,A(k) = A(k-1) + A(k+1). \tag{3.5}$$

The plane-wave parametrization (3.4) solves (3.5) provided

$$\epsilon_1 = -2z_1 + z_5 + \bar{z}_5 + \xi + \xi^{-1}. \tag{3.6}$$

Thus Ψ_1 is the eigenstate of H in the sector $r = 1$ with eigenvalue $E_1 = (N - 2z_1) + z_5 + \bar{z}_5 + 2\cos\theta$, where $\theta = 2\pi l/N$, $l = 0, 1, \ldots, N-1$.

3.3. Sector $r = 2$

In the Hilbert space $\mathcal{H}_2^{(N)}$ we have N states of the type $|k[-]\rangle = \Big|{+}\cdots{+}\underset{k}{-}{+}\cdots{+}\Big\rangle$ and $N(N-1)/2$ states of the type $|k_1[0], k_2[0]\rangle = \Big|{+}\cdots{+}\underset{k_1}{0}{+}\cdots{+}\underset{k_2}{0}{+}\cdots{+}\Big\rangle$. We seek these eigenstates in the form

$$\Psi_2 = \sum_{k_1<k_2} A(k_1,k_2)\,|k_1[0], k_2[0]\rangle + \sum_{k=1}^{N} B(k)\,|k[-]\rangle. \tag{3.7}$$

The periodicity condition now reads

$$A(k_2, N+k_1) = A(k_1,k_2) \qquad \text{and} \qquad B(N+k) = B(k). \tag{3.8}$$

Following Bethe [3], the wavefunction $A(k_1,k_2)$ can be parametrized using the superposition of plane waves (3.4) including the scattering of two *pseudoparticles* with *momenta* θ_1 and θ_2, ($\xi_j = \mathrm{e}^{\mathrm{i}\theta_j}$, $j = 1, 2$):

$$A(k_1,k_2) = A_{12}\xi_1^{k_1}\xi_2^{k_2} + A_{21}\xi_1^{k_2}\xi_2^{k_1} \tag{3.9}$$

which satisfy the periodic boundary condition (3.8) provided

$$A_{12} = A_{21}\xi_1^N \qquad A_{21} = A_{12}\xi_2^N \tag{3.10}$$

and the parametrization of $B(k)$ is still undetermined at this stage.

Before we try to parametrize $B(k)$ let us consider the Schrödinger equation $(E_2 - Nz_1)\,\Psi_2 = \epsilon_2\Psi_2$. From the explicit form of H acting on two sites (2.13) we derive the following set of eigenvalue equations:

$$\begin{aligned}(\epsilon_2 + 4z_1 - 2z_5 - 2\bar{z}_5)\,A(k_1,k_2) &= A(k_1-1,k_2) + A(k_1+1,k_2)\\ &\quad + A(k_1,k_2-1) + A(k_1,k_2+1)\end{aligned} \tag{3.11}$$

$$\begin{aligned}(\epsilon_2 + 3z_1 - z_5 - \bar{z}_5 - \varepsilon z_4)\,A(k,k+1) &= A(k-1,k+1) + A(k,k+2)\\ &\quad + \varepsilon\bar{z}_6\,B(k+1) + \varepsilon z_6\,B(k)\end{aligned} \tag{3.12}$$

$$(\epsilon_2 + 2z_1 - z_7 - \bar{z}_7)\,B(k) = z_3B(k-1) + z_3B(k+1) + \bar{z}_6A(k-1,k) + z_6A(k,k+1). \tag{3.13}$$

The parametrization (3.9) solves the equations (3.11) provided

$$\epsilon_2 = -4z_1 + 2z_5 + 2\bar{z}_5 + \xi_1 + \xi_1^{-1} + \xi_2 + \xi_2^{-1}. \tag{3.14}$$

It follows immediately that the eigenvalues of H are a sum of single pseudoparticle energies.

The parametrization of $B(k)$ can now be determined in the following way: subtracting equation (3.12) from equation (3.11) for $k_1 = k, k_2 = k+1$, we obtain a *meeting equation*

$$\varepsilon\bar{z}_6\, B(k+1) + \varepsilon z_6\, B(k) = A(k,k) + A(k+1,k+1) - (z_1 + \varepsilon z_4 - z_5 - \bar{z}_5)\, A(k,k+1). \tag{3.15}$$

Now we extend the parametrization (3.9) to $k_1 = k_2$ in order to obtain a parametrization for the wavefunction $B(k)$:

$$B(k) = B(\xi_1\xi_2)^k \tag{3.16}$$

which solves the meeting equation (3.15) provided

$$\begin{aligned} B &= \varepsilon\frac{1+\xi_1\xi_2 - \Delta_1\xi_2}{z_6 + \bar{z}_6 x i_1 \xi_2} A_{12} + \varepsilon\frac{1+\xi_1\xi_2 - \Delta_1\xi_1}{z_6 + \bar{z}_6\xi_1\xi_2} A_{21} \\ \Delta_1 &= z_1 + \varepsilon z_4 - z_5 - \bar{z}_5. \end{aligned} \tag{3.17}$$

These relations tell us that the pseudoparticle of the type $|k[-]\rangle$ behaves as the two pseudoparticles $|k_1[0]\rangle$ and $|k_2[0]\rangle$ at the same site k and its parametrization follows as the plane waves of particles $|k_i[0]\rangle$ multiplied by the weight function $B = B(\xi_1, \xi_2)$.

Now substituting (3.9), (3.14) and (3.16) into equation (3.13) we find the phase shift of two pseudoparticles,

$$\frac{A_{21}}{A_{12}} \equiv \Phi_{12} = -\frac{(1+\xi_1\xi_2)^2 - (1+\xi_1\xi_2)(\Delta_2\xi_1 + \Delta_3\xi_2) + \Delta_4\xi_1\xi_2 + \Delta_5\xi_2^2}{(1+\xi_1\xi_2)^2 - (1+\xi_1\xi_2)(\Delta_2\xi_2 + \Delta_3\xi_1) + \Delta_4\xi_1\xi_2 + \Delta_5\xi_1^2} \tag{3.18}$$

where

$$\begin{aligned} \Delta_2 &= \frac{1}{z_3} \\ \Delta_3 &= \frac{1}{z_3} + \frac{\varepsilon}{z_3}(z_3z_4 - z_6\bar{z}_6) + (z_1 - z_5 - \bar{z}_5) \\ \Delta_4 &= \frac{1}{z_3}(\varepsilon z_4 + z_7 + \bar{z}_7) + \frac{3}{z_3}(z_1 - \bar{z}_5 - z_5) - 2 \\ \Delta_5 &= \frac{1}{z_3}(\varepsilon z_4 + z_1 - \bar{z}_5 - z_5). \end{aligned} \tag{3.19}$$

Combining this result with the periodic relations (3.10) and using (2.15)–(2.17) we arrive at the BA equations in $\mathcal{H}_2^{(N)}$ for each model:

$$\xi_2^N = -\left(\frac{1+\xi_1\xi_2 + \xi_1 + \xi_2 - (\Delta+2)\xi_2}{1+\xi_1\xi_2 + \xi_1 + \xi_2 - (\Delta+2)\xi_1}\right) \tag{3.20}$$

for the ZF model,

$$\xi_2^N = -\left(\frac{1+\xi_1\xi_2 - \Delta\xi_2}{1+\xi_1\xi_2 - \Delta\xi_1}\right)\left(\frac{1+\xi_1\xi_2 - \xi_1 - \xi_2 + (\Delta-2)\xi_1}{1+\xi_1\xi_2 - \xi_1 - \xi_2 + (\Delta-2)\xi_2}\right) \tag{3.21}$$

for the IK model and for the $osp(1|2)$ model we obtain

$$\xi_2^N = -\left(\frac{1+\xi_1\xi_2 - \Delta\xi_2}{1+\xi_1\xi_2 - \Delta\xi_1}\right)\left(\frac{1+\xi_1\xi_2 + \xi_1 + \xi_2 - (\Delta+2)\xi_1}{1+\xi_1\xi_2 + \xi_1 + \xi_2 - (\Delta+2)\xi_2}\right) \tag{3.22}$$

where

$$\Delta = 2\cosh 2\eta \qquad \text{and} \qquad (\xi_1\xi_2)^N = 1. \tag{3.23}$$

3.4. Sector $r = 3$

Now the Hilbert space is $\mathcal{H}_3^{(N)}$ where there are $N(N-1)(N-2)/6$ states of the type $|k_1[0], k_2[0], k_3[0]\rangle$, $N(N-1)/2$ states of the type $|k_1[-], k_2[0]\rangle$ and $N(N-1)/2$ states of the type $|k_1[0], k_2[-]\rangle$. We seek eigenfunctions of the form

$$\Psi_3 = \sum_{k_1<k_2<k_3} A(k_1,k_2,k_3)\,|k_1[0],k_2[0],k_3[0]\rangle + \sum_{k<k}\{B_1(k_1,k_2)\,|k_1[-],k_2[0]\rangle + B_2(k_1,k_2)\,|k_1[0],k_2[-]\rangle\}. \tag{3.24}$$

Periodic boundary conditions now read as

$$A(k_2,k_3,N+k_1) = A(k_1,k_2,k_3) \qquad B_2(k_2,N+k_1) = B_1(k_1,k_2). \tag{3.25}$$

Again, the wavefunctions $A(k_1,k_2,k_3)$ can be parametrized by the superposition of plane waves

$$A(k_1,k_2,k_3) = A_{123}\xi_1^{k_1}\xi_2^{k_2}\xi_3^{k_3} + A_{132}\xi_1^{k_1}\xi_2^{k_3}\xi_3^{k_2} + A_{213}\xi_1^{k_2}\xi_2^{k_1}\xi_3^{k_3} + A_{231}\xi_1^{k_2}\xi_2^{k_3}\xi_3^{k_1} + A_{312}\xi_1^{k_3}\xi_2^{k_1}\xi_3^{k_2} + A_{123}\xi_1^{k_3}\xi_2^{k_2}\xi_3^{k_1} \tag{3.26}$$

which satisfy the periodic boundary condition provided

$$\frac{A_{231}}{A_{123}} = \frac{A_{321}}{A_{213}} = \xi_3^N \qquad \frac{A_{213}}{A_{132}} = \frac{A_{312}}{A_{231}} = \xi_2^N \qquad \frac{A_{123}}{A_{312}} = \frac{A_{132}}{A_{321}} = \xi_1^N. \tag{3.27}$$

These relations tell us that the interchange of two pseudoparticles is independent of the position of the third pseudoparticle. Using S-matrix language, this locality of the interactions is equivalent to the factorization property of the S-matrix, according to which the scattering amplitude of three particles factorizes into a product of three two-point S-matrices.

Action of H on these eigenstates gives the following set of coupled equations for $A(k_1,k_2,k_3)$ and $B_i(k_1,k_2)$, $i = 1, 2$:

$$(\epsilon_3 + 6z_1 - 3z_5 - 3\bar{z}_5)\,A(k_1,k_2,k_3) = A(k_1-1,k_2,k_3) + A(k_1+1,k_2,k_3) + A(k_1,k_2-1,k_3) + A(k_1,k_2+1,k_3) + A(k_1,k_2,k_3-1) + A(k_1,k_2,k_3+1). \tag{3.28}$$

These equations show us the action of H in configurations of the Hilbert space $\mathcal{H}_3^{(N)}$ for which the three pseudoparticles ($|k_i[0]\rangle$, $i = 1, 2, 3$) are separated. We already know that they are satisfied with the plane-wave parametrization (3.26) and that

$$\epsilon_3 = \sum_{j=1}^{3}(-2z_1 + z_5 + \bar{z}_5 + \xi_j + \xi_j^{-1}). \tag{3.29}$$

For configurations where two pseudoparticles are neighbours at k_1 and the third pseudoparticle is at $k_2 > k_1 + 2$, H gives us the following equations:

$$(\epsilon_3 + 5z_1 - 2z_5 - 2\bar{z}_5 - \varepsilon z_4)\,A(k_1,k_1+1,k_2) = A(k_1-1,k_1+1,k_2) + A(k_1,k_1+2,k_2) + A(k_1,k_1+1,k_2-1) + A(k_1,k_1+1,k_2+1) + \varepsilon\bar{z}_6\,B_1(k_1+1,k_2) + \varepsilon z_6\,B_1(k_1,k_2) \tag{3.30}$$

and a similar set of equations coupling $B_2(k_1,k_2)$ and $A(k_1,k_2,k_3)$, which correspond to the meeting of two pseudoparticles on the right-hand side of the third pseudoparticle.

Comparing equation (3.30) with equation (3.28) we obtain a consistency equation

$$\varepsilon\bar{z}_6\,B_1(k_1+1,k_2) + \varepsilon z_6\,B_1(k_1,k_2) = A(k_1,k_1,k_2) + A(k_1+1,k_1+1,k_2) - \Delta_1\,A(k_1,k_1+1,k_2). \tag{3.31}$$

Similarly, for the right-hand side meeting we obtain

$$\varepsilon\bar{z}_6\, B_2(k_1, k_2+1) + \varepsilon z_6\, B_2(k_1, k_2) = A(k_1, k_2, k_2) + A(k_1, k_2+1, k_2+1) \\ -\Delta_1\, A(k_1, k_1+1, k_2). \tag{3.32}$$

These consistency equations are solved by the following parametrization of the wavefunctions $B_i(k_1, k_2)$, $i = 1, 2$:

$$B_1(k_1, k_2) = B_{11}(\xi_1\xi_2)^{k_1}\xi_3^{k_2} + B_{12}(\xi_1\xi_3)^{k_1}\xi_2^{k_2} + B_{13}(\xi_2\xi_3)^{k_1}\xi_1^{k_2} \\ B_2(k_1, k_2) = B_{21}\xi_1^{k_1}(\xi_2\xi_3)^{k_2} + B_{22}\xi_2^{k_1}(\xi_1\xi_3)^{k_2} + B_{23}\xi_3^{k_1}(\xi_1\xi_2)^{k_2} \tag{3.33}$$

which satisfy the periodic boundary condition provided

$$B_{21} = \xi_1^N B_{13} \qquad B_{22} = \xi_2^N B_{12} \qquad B_{23} = \xi_3^N B_{11}. \tag{3.34}$$

Moreover, the weight functions B_{1i} and B_{2i}, $i = 1, 2, 3$ are determined

$$B_{11} = F_{12}A_{123} + F_{21}A_{213} \qquad B_{21} = F_{23}A_{123} + F_{32}A_{132} \\ B_{12} = F_{13}A_{132} + F_{31}A_{231} \qquad B_{22} = F_{13}A_{213} + F_{31}A_{312} \\ B_{13} = F_{23}A_{312} + F_{23}A_{321} \qquad B_{23} = F_{12}A_{231} + F_{21}A_{321} \tag{3.35}$$

where

$$F_{ab} = \varepsilon\frac{1 + \xi_a\xi_b - \Delta_1\xi_b}{z_6 + \bar{z}_6\xi_a\xi_b} \qquad a \neq b = 1, 2, 3. \tag{3.36}$$

Substituting these relations into the eigenvalue equations (3.30) we obtain the phase shift of two pseudoparticles

$$\frac{A_{123}}{A_{213}} = \frac{A_{231}}{A_{321}} = \Phi_{12} \qquad \frac{A_{132}}{A_{231}} = \frac{A_{213}}{A_{312}} = \Phi_{13} \qquad \frac{A_{312}}{A_{321}} = \frac{A_{123}}{A_{132}} = \Phi_{23} \tag{3.37}$$

where

$$\Phi_{ab} = -\frac{(1 + \xi_a\xi_b)^2 - (1 + \xi_a\xi_b)(\Delta_2\xi_a + \Delta_3\xi_b) + \Delta_4\xi_a\xi_b + \Delta_5\xi_b^2}{(1 + \xi_a\xi_b)^2 - (1 + \xi_a\xi_b)(\Delta_2\xi_b + \Delta_3\xi_a) + \Delta_4\xi_a\xi_b + \Delta_5\xi_a^2} \qquad a \neq b = 1, 2, 3 \tag{3.38}$$

and the Δ_i, $i = 1, 2, 3, 4$ are given by (3.17) and (3.19)

Next, when the three pseudoparticles are neighbours we have the following eigenvalue equations:

$$(\epsilon_3 + 4z_1 - z_5 - \bar{z}_5 - 2\varepsilon z_4)\, A(k, k+1, k+2) = A(k-1, k+1, k+2) + A(k, k+1, k+3) \\ +\varepsilon\bar{z}_6\, B_1(k+1, k+2) + \varepsilon z_6\, B_1(k, k+2) + \varepsilon\bar{z}_6\, B_2(k, k+2) + \varepsilon z_6\, B_2(k, k+1) \tag{3.39}$$

which are automatically satisfied by the above parametrizations.

In addition to these equations we also have to consider the equations for configurations where the pseudoparticle of type $|k[-]\rangle$ and the pseudoparticle $|k[0]\rangle$ are separated:

$$(\epsilon_3 + 4z_1 - z_5 - \bar{z}_5 - \bar{z}_7 - z_7)\, B_1(k_1, k_2) = B_1(k_1, k_2 - 1) + B_1(k_1, k_2 + 1) \\ +z_3 B_1(k_1 - 1, k_2) + z_3 B_1(k_1 + 1, k_2) + \bar{z}_6 A(k_1 - 1, k_1, k_2) \\ +z_6 A(k_1 + 1, k_1, k_2) \tag{3.40}$$

and a similar set of eigenvalue equations involving $B_2(k_1, k_2)$, which corresponds to configurations with the pseudoparticle $|k_2[-]\rangle$ on the right-hand side of the pseudoparticle $|k_1[0]\rangle$.

These equations are also satisfied by the above parametrizations. This statement was already expected since at this point we always have a *far* particle as a *viewer*. Therefore, none of the configurations appeared to be different from those presented in the sector $r = 2$.

Finally, the action of H on configurations where the two different pseudoparticles are neighbours results in two more eigenvalue equations:

$$(\epsilon_3 + 3z_1 - \bar{z}_7 - 2\bar{z}_5)\, B_1(k, k+1) = B_1(k, k+2) + B_2(k, k+1) + z_3 B_1(k-1, k+1) + \bar{z}_6 A(k-1, k, k+1) \tag{3.41}$$

and

$$(\epsilon_3 + 3z_1 - z_7 - 2z_5)\, B_2(k, k+1) = B_2(k-1, k+1) + B_1(k, k+1) + z_3 B_2(k, k+2) + z_6 A(k, k+1, k+2). \tag{3.42}$$

Substituting the wavefunction parametrizations for $A(k_1, k_2, k_3)$ and $B_i(k_1, k_2)$ into the equations (3.41) and (3.42) and using the relations (3.27) and (3.37) one can verify that they are indeed satisfied. These results tell us that the meeting of the pseudoparticle $|k[-]\rangle$ with the pseudoparticle $|k[0]\rangle$ can be versed as a meeting of three-pseudoparticle $|k[0]\rangle$.

Compounding (3.37) with the periodic boundary conditions (3.27) we arrive to the BA equations for the sector $r = 3$

$$\xi_a^N = \prod_{b \neq a}^{3} \Phi_{ab} \qquad a = 1, 2, 3 \tag{3.43}$$

which expresses the factorization of the three-pseudoparticle phase shift into the product of two-pseudoparticles phase shifts.

3.5. General sector

The above results can now be generalized. First we observe that in the sector $r > 3$ there are no additional meeting conditions. For example, in the sector $r = 4$ there is a meeting of two pseudoparticles of the type $|k[-]\rangle$. Nevertheless, we know that the state $|k[-]\rangle$ is parametrized as two states $|k[0]\rangle$ at the same site and we have verified that the meeting of two pseudoparticles $|k[-]\rangle$ behaves as the meeting of four pseudoparticles $|k[0]\rangle$ whose phase shift factorizes in a product of two-pseudoparticles phase shifts.

In a generic sector r we build eigenstates of H out of translationally invariant products of N_0 one-pseudoparticle eigenstates $|k[0]\rangle$ and N_- two-pseudoparticle eigenstates $|k[-]\rangle$, such that $r = N_0 + 2N_-$. These eigenstates are obtained by superposition of terms of the form

$$|\phi_r\rangle = |0\rangle \times |\phi_{r-1}\rangle + |-\rangle \times |\phi_{r-2}\rangle \tag{3.44}$$

with $|\phi_0\rangle = 1$, $|\phi_1\rangle = |0\rangle$. The corresponding eigenvalue is a sum of single one-particle energies

$$E_r = N z_1 + \sum_{a=1}^{r} \left(-2z_1 + z_5 + \bar{z}_5 + \xi_a + \xi_a^{-1}\right) \tag{3.45}$$

being ξ_a solutions of the BA equations

$$\xi_a^N = \prod_{b \neq a=1}^{r} \Phi_{ab} \qquad a = 1, 2, \ldots, r \tag{3.46}$$

where

$$\Phi_{ab} = -\left(\frac{1 + \xi_a\xi_b + \xi_a + \xi_b - (\Delta + 2)\xi_b}{1 + \xi_a\xi_b + \xi_a + \xi_b - (\Delta + 2)\xi_a}\right) \qquad a, b = 1, 2, \ldots, r \qquad \Delta = 2\cosh 2\eta \tag{3.47}$$

for the ZF model,

$$\Phi_{ab} = -\left(\frac{1+\xi_a\xi_b-\Delta\xi_b}{1+\xi_a\xi_b-\Delta\xi_a}\right)\left(\frac{1+\xi_a\xi_b-\xi_a-\xi_b+(\Delta-2)\xi_a}{1+\xi_a\xi_b-\xi_a-\xi_b+(\Delta-2)\xi_b}\right) \qquad (3.48)$$
$$a, b = 1, 2, \ldots, r \qquad \Delta = 2\cosh 2\eta$$

for the IK model and

$$\Phi_{ab} = -\left(\frac{1+\xi_a\xi_b-\Delta\xi_b}{1+\xi_a\xi_b-\Delta\xi_a}\right)\left(\frac{1+\xi_a\xi_b+\xi_a+\xi_b-(\Delta+2)\xi_a}{1+\xi_a\xi_b+\xi_a+\xi_b-(\Delta+2)\xi_b}\right) \qquad (3.49)$$
$$a, b = 1, 2, \ldots, r \qquad \Delta = 2\cosh 2\eta$$

for the $osp(1|2)$ model.

4. The algebraic Bethe ansatz

In the previous section we have considered the problem of diagonalization of a one-dimensional spin-chain Hamiltonian using the coordinate BA. Let us now turn to two-dimensional classical statistical systems on a lattice.

Let us consider a regular lattice with N columns and N' rows. A physical state on this lattice is defined by the assignment of a *state variable* to each lattice edge. If one takes the horizontal direction as space and the vertical one as time, the transfer matrix plays the role of a discrete evolution operator acting on the Hilbert space $\mathcal{H}^{(N)}$ spanned by the *row states* which are defined by the set of vertical link variables on the same row. Thus, the transfer matrix elements can be understood as the transition probability of the one-row state to project on the consecutive one after a unit of time.

The main problem now is the diagonalization of the transfer matrix of the lattice system. To do this we request the algebraic BA.

Again, we start with the graded formulation such that the additional signs are represented by $\varepsilon = -1$. Taking $\varepsilon = 1$ we recover the non-graded cases.

We recall some basic relations of the graded quantum inverse scattering method. For us the basic object will be the R-matrix (2.6), which satisfies $R(0,\eta) = \rho(\eta)\mathcal{I}$, where $\rho_{\mathrm{ZF}}(\eta) = \sinh\eta\sinh 2\eta$, $\rho_{\mathrm{IK}}(\eta) = -\sinh 5\eta + \sinh\eta$ and $\rho_{\mathrm{osp}}(\eta) = \sinh 2\eta\sinh 3\eta$.

A quantum-integrable system is characterized by monodromy matrix $T(\lambda)$ satisfying the equation

$$R(\lambda-\mu)\big[T(\lambda)\overset{s}{\otimes}T(\mu)\big] = \big[T(\mu)\overset{s}{\otimes}T(\lambda)\big]R(\lambda-\mu) \qquad (4.1)$$

whose consistency is guaranteed by the YB equation (2.5). $T(\lambda)$ is a matrix in the space V with matrix elements that are operators on the states of the quantum system (the quantum space, which will also be the space V). The space V is called auxiliary space of $T(\lambda)$. An example of a monodromy matrix is the matrix $\mathcal{P}R$, this follows directly from (4.1).

The simplest monodromies have become known as $\mathcal{L}$ operators, the Lax operator, and the monodromy operator $T(\lambda)$ is defined as an ordered product of Lax operators on all sites of the lattice:

$$T(\lambda) = \mathcal{L}_N(\lambda)\,\mathcal{L}_{N-1}(\lambda)\ldots\mathcal{L}_1(\lambda). \qquad (4.2)$$

The Lax operator on the nth quantum space is given the graded permutation of (2.6):

$$\mathcal{L}_n(\lambda) = \begin{pmatrix} x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & x_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & x_6 & 0 & x_7 & 0 & 0 \\ 0 & y_5 & 0 & x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_6 & 0 & x_4 & 0 & x_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_2 & 0 & x_5 & 0 \\ 0 & 0 & y_7 & 0 & y_6 & 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_5 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 \end{pmatrix} = \begin{pmatrix} L_{11}^{(n)}(\lambda) & L_{12}^{(n)}(\lambda) & L_{13}^{(n)}(\lambda) \\ L_{21}^{(n)}(\lambda) & L_{22}^{(n)}(\lambda) & L_{23}^{(n)}(\lambda) \\ L_{31}^{(n)}(\lambda) & L_{32}^{(n)}(\lambda) & L_{33}^{(n)}(\lambda) \end{pmatrix}. \tag{4.3}$$

Note that $L_{ij}^{(n)}(\lambda)$, $i, j = 1, 2, 3$ are 3×3 matrices acting on the nth site of the lattice. It means that the monodromy matrix has the form

$$T(\lambda) = \begin{pmatrix} T_{11}(\lambda) & T_{12}(\lambda) & T_{13}(\lambda) \\ T_{21}(\lambda) & T_{22}(\lambda) & T_{23}(\lambda) \\ T_{31}(\lambda) & T_{32}(\lambda) & T_{33}(\lambda) \end{pmatrix} = \begin{pmatrix} A_1(\lambda) & B_1(\lambda) & B_2(\lambda) \\ C_1(\lambda) & A_2(\lambda) & B_3(\lambda) \\ C_2(\lambda) & C_3(\lambda) & A_3(\lambda) \end{pmatrix} \tag{4.4}$$

where

$$T_{ij}(\lambda) = \sum_{k_1,\ldots,k_{N-1}=1}^{3} L_{ik_1}^{(N)}(\lambda) \overset{s}{\otimes} L_{k_1k_2}^{(N-1)}(\lambda) \overset{s}{\otimes} \cdots \overset{s}{\otimes} L_{k_{N-1}j}^{(1)}(\lambda) \qquad i, j = 1, 2, 3. \tag{4.5}$$

The vector $|0\rangle$ in the quantum space of the monodromy matrix $T(\lambda)$ that is annihilated by the operators $T_{ij}(\lambda)$, $i > j$ ($C_k(\lambda)$ operators, $k = 1, 2, 3$) and is an eigenvector for the operators $T_{ii}(\lambda)$ ($A_k(\lambda)$ operators, $k = 1, 2, 3$) is called a highest vector of the monodromy matrix $T(\lambda)$.

The transfer matrix $\tau(\lambda)$ of the corresponding integrable spin model is given by the supertrace of the monodromy matrix in the space V, Str $T(\lambda)$. It is the generating function of the family of commuting operators in terms of which the Hamiltonian of the quantum system is expressed:

$$\tau(\lambda) = \mathrm{Str}\, T(\lambda) = \sum_{i=1}^{3} (-)^{p(i)}\, T_{ii}(\lambda) = A_1(\lambda) + \varepsilon A_2(\lambda) + A_3(\lambda). \tag{4.6}$$

In particular, the Hamiltonians (2.12) can also be derived by the well known relation

$$H = \alpha \frac{\partial}{\partial \lambda} (\ln \tau(\lambda))_{\lambda=0}. \tag{4.7}$$

A detailed exposition of the graded quantum inverse scattering method can be found in [22].

In this section we will derive the BA equations of 19-vertex models presented in section 2 using the algebraic BA developed by Tarasov [16] and generalized recently by Martins and Ramos [23]. To do this we need the commutation relations for entries of the monodromy matrix which are derived from the fundamental relation (4.1). Here these commutation relations do not share a common structure. Therefore, we only write some of them in the text and recall (4.1) to obtain the remaining ones.

First of all, let us observe that for each row state one can define the magnon number operator which commutes with the transfer matrix of the models

$$[\tau(\lambda), M] = 0 \qquad M = \sum_{k=1}^{N} M_k \qquad M_k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{4.8}$$

This is the analogue of the operator S_T^z used in the previous section and the relation between M and the spin total S_T^z is simply $M = N - S_T^z$. Once again, the Hilbert space can be broken down into sectors $\mathcal{H}_M^{(N)}$. In each of these sectors, the transfer matrix can be diagonalized independently, $\tau(\lambda)\,\Psi_M = \Lambda_M \Psi_M$. We will now start to diagonalize $\tau(\lambda)$ in every sector:

4.1. Sector $M = 0$

Let us consider the highest vector of the monodromy matrix $T(\lambda)$ in a lattice of N sites as the even (bosonic) completely unoccupied state

$$\Psi_0 \equiv |0\rangle = \otimes_{k=1}^{N} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_k. \tag{4.9}$$

It is the only state in the sector with $M = 0$. Using (4.5) we can compute the action of the matrix elements of $T(\lambda)$ on this reference state:

$$\begin{aligned} &A_1(\lambda)\,|0\rangle = x_1^N(\lambda)\,|0\rangle \qquad A_2(\lambda)\,|0\rangle = x_2^N(\lambda)\,|0\rangle \qquad A_3(\lambda)\,|0\rangle = x_3^N(\lambda)\,|0\rangle \\ &C_k(\lambda)\,|0\rangle = 0 \qquad\qquad B_k(\lambda)\,|0\rangle \neq \{0, |0\rangle\} \qquad\quad k = 1, 2, 3. \end{aligned} \tag{4.10}$$

Therefore in the sector $M = 0$, Ψ_0 is the eigenstate of $\tau(\lambda) = A_1(\lambda) + \varepsilon A_2(\lambda) + A_3(\lambda)$ with eigenvalue

$$\Lambda_0(\lambda) = x_1^N(\lambda) + \varepsilon x_2^N(\lambda) + x_3^N(\lambda). \tag{4.11}$$

Here we observe that the action of the operators $B_1(\lambda)$, $B_2(\lambda)$ and $B_3(\lambda)$ on the reference state will give us new states which lie in sectors with $M \neq 0$.

4.2. Sector $M = 1$

In this sector we have the states $B_1(\lambda)\,|0\rangle$ and $B_3(\lambda)\,|0\rangle$. Since $B_3(\lambda)\,|0\rangle \propto B_1(\lambda)\,|0\rangle$, we seek an eigenstate of the form

$$\Psi_1(\lambda_1) = B_1(\lambda_1)\,|0\rangle. \tag{4.12}$$

The action of the operator $\tau(\lambda)$ on this state can be computed with the aid of the following commutation relations:

$$A_1(\lambda)\,B_1(\mu) = z(\mu-\lambda)\,B_1(\mu)\,A_1(\lambda) - \frac{x_5(\mu-\lambda)}{x_2(\mu-\lambda)} B_1(\lambda)\,A_1(\mu) \tag{4.13}$$

$$\begin{aligned} A_2(\lambda)\,B_1(\mu) = {} & \varepsilon \frac{z(\lambda-\mu)}{\omega(\lambda-\mu)} B_1(\mu)\,A_2(\lambda) - \frac{z(\lambda-\mu)}{\omega(\lambda-\mu)} \frac{1}{y(\mu-\lambda)} B_2(\mu)\,C_1(\lambda) \\ & - \varepsilon \frac{y_5(\lambda-\mu)}{x_2(\lambda-\mu)} B_1(\lambda)\,A_2(\mu) + \frac{y_5(\lambda-\mu)}{x_2(\lambda-\mu)} \frac{1}{y(\lambda-\mu)} B_2(\lambda)\,C_1(\mu) \\ & + \frac{1}{y(\lambda-\mu)} B_3(\lambda)\,A_1(\mu) \end{aligned} \tag{4.14}$$

$$\begin{aligned} A_3(\lambda)\,B_1(\mu) = {} & \frac{x_2(\lambda-\mu)}{x_3(\lambda-\mu)} B_1(\mu)\,A_3(\lambda) - \frac{\varepsilon}{y(\lambda-\mu)} B_3(\lambda)\,A_2(\mu) \\ & + \frac{x_5(\lambda-\mu)}{x_3(\lambda-\mu)} B_2(\mu)\,C_3(\lambda) - \frac{y_7(\lambda-\mu)}{x_3(\lambda-\mu)} B_2(\lambda)\,C_3(\mu). \end{aligned} \tag{4.15}$$

The ratio functions which appear in the commutation relations are defined by

$$\begin{aligned} z(\lambda) &= \frac{x_1(\lambda)}{x_2(\lambda)} \qquad \omega(\lambda) = \varepsilon \frac{x_1(\lambda)\, x_3(\lambda)}{x_3(\lambda)\, x_4(\lambda) - x_6(\lambda)\, y_6(\lambda)} \\ y(\lambda) &= \frac{x_3(\lambda)}{y_6(\lambda)} \qquad y(-\lambda) = \varepsilon \frac{x_3(\lambda)\, x_4(\lambda) - x_6(\lambda)\, y_6(\lambda)}{x_7(\lambda)\, y_6(\lambda) - x_3(\lambda)\, x_6(\lambda)}. \end{aligned} \tag{4.16}$$

When $\tau(\lambda)$ act on $\Psi_1(\lambda_1)$, the corresponding eigenvalue equation has two unwanted terms:

$$\begin{aligned} \tau(\lambda)\, \Psi_1(\lambda_1) &= (A_1(\lambda) + \varepsilon A_2(\lambda) + A_3(\lambda))\Psi_1(\lambda_1) \\ &= \left[z(\lambda_1 - \lambda)\, x_1^N(\lambda) + \varepsilon^2 \frac{z(\lambda - \lambda_1)}{\omega(\lambda - \lambda_1)} x_2^N(\lambda) + \frac{x_2(\lambda - \lambda_1)}{x_3(\lambda - \lambda_1)} x_3^N(\lambda) \right] \Psi_1(\lambda_1) \\ &\quad - \left[\frac{x_5(\lambda_1 - \lambda)}{x_2(\lambda_1 - \lambda)} x_1^N(\lambda_1) + \varepsilon^2 \frac{y_5(\lambda - \lambda_1)}{x_2(\lambda - \lambda_1)} x_2^N(\lambda_1) \right] B_1(\lambda)\, |0\rangle \\ &\quad + \varepsilon \left[\frac{1}{y(\lambda - \lambda_1)} x_1^N(\lambda_1) - \frac{1}{y(\lambda - \lambda_1)} x_2^N(\lambda_1) \right] B_3(\lambda)\, |0\rangle. \end{aligned} \tag{4.17}$$

From the matrix elements of 19-vertex models (2.7)–(2.9) we can see that $x_5(\lambda)/x_2(\lambda) = -y_5(-\lambda)/x_2(-\lambda)$. Therefore the unwanted terms vanish and $\Psi_1(\lambda_1)$ is an eigenstate of $\tau(\lambda)$ with eigenvalue

$$\Lambda_1(\lambda, \lambda_1) = z(\lambda_1 - \lambda)\, x_1^N(\lambda) + \varepsilon^2 \frac{z(\lambda - \lambda_1)}{\omega(\lambda - \lambda_1)} x_2^N(\lambda) + \frac{x_2(\lambda - \lambda_1)}{x_3(\lambda - \lambda_1)} x_3^N(\lambda) \tag{4.18}$$

provided

$$\left(\frac{x_1(\lambda_1)}{x_2(\lambda_1)} \right)^N = \varepsilon^2 = 1. \tag{4.19}$$

4.3. Sector $M = 2$

In the sector $M = 2$, we encounter two linearly independent states $B_1(\lambda)\, B_1(\mu)\, |0\rangle$ and $B_2(\lambda)\, |0\rangle$. (The states $B_3 B_3\, |0\rangle$, $B_1 B_3\, |0\rangle$ and $B_3 B_1\, |0\rangle$ also lie in the sector $M = 2$ but they are proportional to the state $B_1 B_1\, |0\rangle$). We seek eigenstates in the form

$$\Psi_2(\lambda_1, \lambda_2) = B_1(\lambda_1)\, B_1(\lambda_2)\, |0\rangle + B_2(\lambda_1)\, \Gamma(\lambda_1, \lambda_2)\, |0\rangle \tag{4.20}$$

where $\Gamma(\lambda_1, \lambda_2)$ is an operator-valued function which has to be fixed such that $\Psi_2(\lambda_1, \lambda_2)$ is unique state in the sector $M = 2$.

Here we observe that the operator-valued function $\Gamma(\lambda_1, \lambda_2)$ is the analogue of the weight function $B(\xi_1, \xi_2)$ of equation (3.17).

It was demonstrated in [16] that $\Psi_2(\lambda_1, \lambda_2)$ is unique provided it is ordered in a normal way: in general, the operator-valued function $\Psi_n(\lambda_1, \ldots, \lambda_n)$ is a composite of normal-ordered monomials. A monomial is normally ordered if in it all elements of the type $B_i(\lambda)$ are on the left, and all elements of the type $C_j(\lambda)$ are on the right of all elements of the type $A_k(\lambda)$. Moreover, the elements of one given type have standard ordering: $T_{i_1 j_1}(\lambda_1)\, T_{i_2 j_2}(\lambda_2) \ldots T_{i_n jn}(\lambda_n)$. For a given sector $M = n$, $\Psi_n(\lambda_1, \ldots, \lambda_n)$ is unique.

From the commutation relation

$$B_1(\lambda)\, B_1(\mu) = \omega(\mu - \lambda) \left[B_1(\mu)\, B_1(\lambda) - \frac{1}{y(\mu - \lambda)} B_2(\mu)\, A_1(\lambda) \right] + \frac{1}{y(\lambda - \mu)} B_2(\lambda)\, A_1(\mu) \tag{4.21}$$

we can see that (4.21) will be normally ordered if it satisfies the following swap condition:

$$\Psi_2(\lambda_2, \lambda_1) = \omega(\lambda_1 - \lambda_2)\, \Psi_2(\lambda_1, \lambda_2). \tag{4.22}$$

This condition fixes $\Gamma(\lambda_1, \lambda_2)$ in equation (4.20) and the eigenstate of $\tau(\lambda)$ in the sector $M = 2$ has the form

$$\Psi_2(\lambda_1, \lambda_2) = B_1(\lambda_1)\, B_1(\lambda_2)\, |0\rangle - \frac{1}{y(\lambda_1 - \lambda_2)} B_2(\lambda_1)\, A_1(\lambda_2)\, |0\rangle. \tag{4.23}$$

The action of a transfer matrix on the states of the form (4.23) is more laborious. In addition to (4.13)–(4.15) and (4.21) we need appeal to (4.1) to derive eight more commutation relations:

$$\begin{aligned} A_1(\lambda)\, B_2(\mu) &= \frac{x_1(\mu-\lambda)}{x_3(\mu-\lambda)} B_2(\mu)\, A_1(\lambda) - \frac{x_7(\mu-\lambda)}{x_3(\mu-\lambda)} B_2(\lambda)\, A_1(\mu) \\ &\quad - \varepsilon \frac{x_6(\mu-\lambda)}{x_3(\mu-\lambda)} B_1(\lambda)\, B_1(\mu) \end{aligned} \tag{4.24}$$

$$\begin{aligned} A_2(\lambda)\, B_2(\mu) &= z(\lambda-\mu)\, z(\mu-\lambda)\, B_2(\mu)\, A_2(\lambda) \\ &\quad + \frac{y_5(\lambda-\mu)}{x_2(\lambda-\mu)} \left[B_1(\lambda)\, B_3(\mu) - \varepsilon B_3(\lambda)\, B_1(\mu) + \frac{y_5(\lambda-\mu)}{x_2(\lambda-\mu)} B_2(\lambda)\, A_2(\mu) \right] \end{aligned} \tag{4.25}$$

$$\begin{aligned} A_3(\lambda)\, B_2(\mu) &= \frac{x_1(\lambda-\mu)}{x_3(\lambda-\mu)} B_2(\mu)\, A_3(\lambda) - \frac{y_7(\lambda-\mu)}{x_3(\lambda-\mu)} B_2(\lambda)\, A_3(\mu) \\ &\quad - \frac{\varepsilon}{y(\lambda-\mu)} B_3(\lambda)\, B_3(\mu) \end{aligned} \tag{4.26}$$

$$C_1(\lambda)\, B_1(\mu) = \varepsilon B_1(\mu)\, C_1(\lambda) + \frac{y_5(\lambda-\mu)}{x_2(\lambda-\mu)} [A_1(\mu)\, A_2(\lambda) - A_1(\lambda)\, A_2(\mu)] \tag{4.27}$$

$$\begin{aligned} C_3(\lambda)\, B_1(\mu) &= \varepsilon \frac{x_4(\lambda-\mu)}{x_3(\lambda-\mu)} B_1(\mu)\, C_3(\lambda) - \frac{x_7(\lambda-\mu)}{x_3(\lambda-\mu)} B_1(\lambda)\, C_3(\mu) \\ &\quad + \frac{1}{y(\lambda-\mu)} [A_1(\mu)\, A_3(\lambda) - A_2(\lambda)\, A_2(\mu)] + \frac{x_6(\lambda-\mu)}{x_3(\lambda-\mu)} B_2(\mu)\, C_2(\lambda) \end{aligned} \tag{4.28}$$

$$B_1(\lambda)\, B_2(\mu) = \frac{1}{z(\lambda-\mu)} B_2(\mu)\, B_1(\lambda) + \frac{y_5(\lambda-\mu)}{x_1(\lambda-\mu)} B_1(\mu)\, B_2(\lambda) \tag{4.29}$$

$$B_1(\lambda)\, B_3(\mu) = \varepsilon B_3(\mu)\, B_1(\lambda) - \frac{y_5(\lambda-\mu)}{x_2(\lambda-\mu)} B_2(\mu)\, A_2(\lambda) + \frac{x_5(\lambda-\mu)}{x_2(\lambda-\mu)} B_2(\lambda)\, A_2(\mu) \tag{4.30}$$

$$B_2(\lambda)\, B_1(\mu) = \frac{1}{z(\lambda-\mu)} B_1(\mu)\, B_2(\lambda) + \frac{x_5(\lambda-\mu)}{x_1(\lambda-\mu)} B_2(\mu)\, B_1(\lambda). \tag{4.31}$$

Here we observe that in this approach the final action of $\tau(\lambda)$ on normally ordered states must be normal ordered. This implies an increasing use of commutation relations needed for the diagonalization of $\tau(\lambda)$. For example, the action of the operator $A_1(\lambda)$ on $\Psi_2(\lambda_1, \lambda_2)$ has the form

$$\begin{aligned} A_1(\lambda)\, \Psi_2(\lambda_1, \lambda_2) &= z(\lambda_{10})\, z(\lambda_{20})\, x_1^N(\lambda)\, \Psi_2(\lambda_1, \lambda_2) - \frac{x_5(\lambda_{10})}{x_2(\lambda_{10})} z(\lambda_{21})\, x_1^N(\lambda_1)\, B_1(\lambda)\, B_1(\lambda_2)\, |0\rangle \\ &\quad - \frac{x_5(\lambda_{20})}{x_2(\lambda_{20})} \frac{z(\lambda_{12})}{\omega(\lambda_{12})} x_1^N(\lambda_2)\, B_1(\lambda)\, B_1(\lambda_1)\, |0\rangle \\ &\quad + \left(\frac{z(\lambda_{10})}{\omega(\lambda_{10})} \frac{x_5(\lambda_{20})}{x_2(\lambda_{20})} \frac{1}{y(\lambda_{01})} + \frac{x_7(\lambda_{10})}{x_3(\lambda_{10})} \frac{1}{y(\lambda_{12})} \right) x_1^N(\lambda_1)\, x_1^N(\lambda_2)\, B_2(\lambda)\, |0\rangle \end{aligned} \tag{4.32}$$

where $\lambda_{ab} = \lambda_a - \lambda_b$, $a \neq b = 0, 1, 2$, with $\lambda_0 = \lambda$. Here we have used the following identities satisfied by the matrix elements of these 19-vertex models:

$$\begin{aligned}
&\frac{z(\lambda_{ab})}{\omega(\lambda_{ab})}\frac{x_5(\lambda_{cb})}{x_2(\lambda_{cb})} - \varepsilon\frac{x_6(\lambda_{ab})}{x_3(\lambda_{ab})}\frac{1}{y(\lambda_{ac})} = \frac{x_5(\lambda_{ab})}{x_2(\lambda_{ab})}\frac{x_5(\lambda_{ca})}{x_2(\lambda_{ca})} + \frac{z(\lambda_{ac})}{\omega(\lambda_{ac})}\frac{x_5(\lambda_{cb})}{x_2(\lambda_{cb})} \\
&z(\lambda_{ab})\frac{x_5(\lambda_{cb})}{x_2(\lambda_{cb})}\frac{1}{y(\lambda_{ab})} + \frac{x_1(\lambda_{ab})}{x_3(\lambda_{ab})}\frac{1}{y(\lambda_{ac})} = z(\lambda_{ab})\, z(\lambda_{cb})\frac{1}{y(\lambda_{ac})} \\
&\omega(\lambda_{ab})\,\omega(\lambda_{ba}) = 1 \qquad\qquad (a \neq b \neq c).
\end{aligned} \tag{4.33}$$

Similarly, for the operator $A_2(\lambda)$ we have

$$\begin{aligned}
A_2(\lambda)\,\Psi_2(\lambda_1,\lambda_2) = {}& \varepsilon^2\frac{z(\lambda_{01})}{\omega(\lambda_{01})}\frac{z(\lambda_{02})}{\omega(\lambda_{02})}x_2^N(\lambda)\,\Psi_2(\lambda_1,\lambda_2) \\
&-\varepsilon^2\frac{y_5(\lambda_{02})}{x_2(\lambda_{02})}z(\lambda_{21})\,x_2^N(\lambda_2)\,B_1(\lambda)\,B_1(\lambda_1)\,|0\rangle \\
&-\varepsilon^2\frac{y_5(\lambda_{01})}{x_2(\lambda_{01})}\frac{z(\lambda_{12})}{\omega(\lambda_{12})}x_2^N(\lambda_1)\,B_1(\lambda)\,B_1(\lambda_2)\,|0\rangle \\
&+z(\lambda_{21})\frac{1}{y(\lambda_{01})}x_1^N(\lambda_1)\,B_3(\lambda)\,B_1(\lambda_2)\,|0\rangle \\
&+\frac{z(\lambda_{12})}{\omega(\lambda_{12})}\frac{1}{y(\lambda_{02})}x_1^N(\lambda_2)\,B_3(\lambda)\,B_1(\lambda_1)\,|0\rangle \\
&+\varepsilon^2\frac{y_5(\lambda_{01})}{x_2(\lambda_{01})}\left(\frac{y_5(\lambda_{21})}{x_2(\lambda_{21})}\frac{1}{y(\lambda_{01})} + \frac{z(\lambda_{01})}{\omega(\lambda_{01})}\frac{1}{y(\lambda_{02})} - \frac{y_5(\lambda_{01})}{x_2(\lambda_{01})}\frac{1}{y(\lambda_{12})}\right) \\
&\times x_1^N(\lambda_2)\,x_2^N(\lambda_1)\,B_2(\lambda)\,|0\rangle \\
&+\varepsilon^2\frac{1}{y(\lambda_{01})}\left(z(\lambda_{01})\frac{y_5(\lambda_{02})}{x_2(\lambda_{02})} - \frac{y_5(\lambda_{01})}{x_2(\lambda_{01})}\frac{y_5(\lambda_{02})}{x_2(\lambda_{02})}\right)x_1^N(\lambda_1)\,x_2^N(\lambda_2)\,B_2(\lambda)\,|0\rangle.
\end{aligned} \tag{4.34}$$

In this case we have used more two identities:

$$\begin{aligned}
&\frac{z(\lambda_{ab})}{\omega(\lambda_{ab})}\frac{1}{y(\lambda_{ac})} + \frac{y_5(\lambda_{cb})}{x_2(\lambda_{cb})}\frac{1}{y(\lambda_{ab})} = \frac{y_5(\lambda_{ab})}{x_2(\lambda_{ab})}\frac{1}{y(\lambda_{bc})} + \frac{z(\lambda_{bc})}{\omega(\lambda_{bc})}\frac{1}{y(\lambda_{ac})} \\
&z(\lambda_{cb})\frac{y_5(\lambda_{ac})}{x_2(\lambda_{ac})} + \frac{y_5(\lambda_{ab})}{x_2(\lambda_{ab})}\frac{y_5(\lambda_{bc})}{x_2(\lambda_{bc})} = \frac{z(\lambda_{ab})\,y_5(\lambda_{ac})}{x_2(\lambda_{ac})} \\
&a \neq b \neq c.
\end{aligned} \tag{4.35}$$

Finally, for $A_3(\lambda)$ we obtain

$$\begin{aligned}
A_3(\lambda)\,\Psi_2(\lambda_1,\lambda_2) = {}& \frac{x_2(\lambda_{01})}{x_3(\lambda_{01})}\frac{x_2(\lambda_{02})}{x_3(\lambda_{02})}x_3^N(\lambda)\,\Psi_2(\lambda_1,\lambda_2) \\
&-\varepsilon^2\frac{z(\lambda_{12})}{\omega(\lambda_{12})}\frac{1}{y(\lambda_{01})}x_2^N(\lambda_1)\,B_3(\lambda)\,B_1(\lambda_2)\,|0\rangle \\
&-\varepsilon^2 z(\lambda_{21})\frac{1}{y(\lambda_{02})}x_2^N(\lambda_2)\,B_3(\lambda)\,B_1(\lambda_1)\,|0\rangle \\
&+\left(\frac{y_7(\lambda_{01})}{x_3(\lambda_{01})}\frac{1}{y(\lambda_{12})} - \frac{y_5(\lambda_{01})}{x_3(\lambda_{01})}\frac{1}{y(\lambda_{02})}\right)x_2^N(\lambda_1)\,x_2^N(\lambda_2)\,B_2(\lambda)\,|0\rangle.
\end{aligned} \tag{4.36}$$

Here we have also used the identities (4.33) and (4.35).

From these relations one can see that all unwanted terms of $\tau(\lambda)\,\Psi_2(\lambda_1,\lambda_2)$ vanish. It means that $\Psi_2(\lambda_1,\lambda_2)$ is an eigenstate of the transfer matrix $\tau(\lambda)$ with eigenvalue

$$\Lambda_2(\lambda,\lambda_1,\lambda_2) = z(\lambda_{10})\,z(\lambda_{20})\,x_1^N(\lambda) + \varepsilon^3\frac{z(\lambda_{01})}{\omega(\lambda_{01})}\frac{z(\lambda_{02})}{\omega(\lambda_{02})}x_2^N(\lambda) + \frac{x_2(\lambda_{01})}{x_3(\lambda_{01})}\frac{x_2(\lambda_{02})}{x_3(\lambda_{02})}x_3^N(\lambda) \tag{4.37}$$

provided the rapidities λ_1 and λ_2 satisfy the BA equations

$$\left(\frac{x_1(\lambda_a)}{x_2(\lambda_a)}\right)^N = \varepsilon^3\frac{z(\lambda_{ab})}{z(\lambda_{ba})}\omega(\lambda_{ba}) \qquad a\neq b=1,2. \tag{4.38}$$

4.4. General sector

The generalization of the above results to sectors with more than two particles proceeds through the factorization properties of the higher-order phase shifts discussed in the previous section. Therefore, at this point we shall present the general result: in a generic sector $M=n$, we have $n-1$ swap conditions

$$\Psi_n(\lambda_1,\ldots,\lambda_{i-1},\lambda_{i+1},\lambda_i,\ldots,\lambda_n) = \omega(\lambda_i-\lambda_{i+1})\,\Psi_n(\lambda_1,\ldots,\lambda_{i-1},\lambda_i,\lambda_{i+1},\ldots,\lambda_n) \tag{4.39}$$

which yield the $n-1$ operator-valued functions $\Gamma_i(\lambda_1,\ldots,\lambda_n)$. The corresponding normal-ordered state $\Psi_n(\lambda_1,\ldots,\lambda_n)$ can be written with the aid of a recurrence formula [16]:

$$\Psi_n(\lambda_1,\ldots,\lambda_n) = \Phi_n(\lambda_1,\ldots,\lambda_n)\,|0\rangle \tag{4.40}$$

where

$$\begin{aligned}\Phi_n(\lambda_1,\ldots,\lambda_n) &= B_1(\lambda_1)\,\Phi_{n-1}(\lambda_2,\ldots,\lambda_n)\\ &\quad -B_2(\lambda_1)\sum_{j=2}^{n}\frac{1}{y(\lambda_1-\lambda_j)}\prod_{k=2,k\neq j}^{n}\mathcal{Z}(\lambda_k-\lambda_j)\,\Phi_{n-2}(\lambda_2,\ldots,\hat{\lambda}_j,\ldots,\lambda_n)\,A_1(\lambda_j)\end{aligned} \tag{4.41}$$

with the initial condition $\Phi_0=1$, $\Phi_1(\lambda)=B_1(\lambda)$.

The scalar function $\mathcal{Z}(\lambda_k-\lambda_j)$ is defined by

$$\mathcal{Z}(\lambda_k-\lambda_j) = \begin{cases} z(\lambda_k-\lambda_j) & \text{if} \quad k>j \\ z(\lambda_k-\lambda_j)\,\omega(\lambda_j-\lambda_k) & \text{if} \quad k<j. \end{cases} \tag{4.42}$$

The action of the operators $A_i(\lambda), i=1,2,3$ on the operators Φ_n have the following normal-ordered form:

$$\begin{aligned}A_1(\lambda)\,\Phi_n(\lambda_1,\ldots,\lambda_n) &= \prod_{k=1}^{n}z(\lambda_k-\lambda)\,\Phi_n(\lambda_1,\ldots,\lambda_n)\,A_1(\lambda)\\ &\quad -B_1(\lambda)\sum_{j=1}^{n}\frac{x_5(\lambda_j-\lambda)}{x_2(\lambda_j-\lambda)}\prod_{k=1,k\neq j}^{n}\mathcal{Z}(\lambda_k-\lambda_j)\,\Phi_{n-1}(\lambda_1,\ldots,\hat{\lambda}_j,\ldots,\lambda_n)\,A_1(\lambda_j)\\ &\quad +B_2(\lambda)\sum_{j=2}^{n}\sum_{l=1}^{j-1}G_{jl}(\lambda,\lambda_l,\lambda_j)\prod_{k=1,k\neq j,l}^{n}\mathcal{Z}(\lambda_k-\lambda_l)\,\mathcal{Z}(\lambda_k-\lambda_j)\\ &\quad \times\Phi_{n-2}(\lambda_1,\ldots,\hat{\lambda}_l,\ldots,\hat{\lambda}_j,\ldots,\lambda_n)\,A_1(\lambda_l)\,A_1(\lambda_j)\end{aligned} \tag{4.43}$$

where $G_{jl}(\lambda,\lambda_l,\lambda_j)$ are scalar functions defined by

$$G_{jl}(\lambda,\lambda_l,\lambda_j) = \frac{x_7(\lambda_l-\lambda)}{x_3(\lambda_l-\lambda)}\frac{1}{y(\lambda_l-\lambda_j)} + \frac{z(\lambda_l-\lambda)}{\omega(\lambda_l-\lambda)}\frac{x_5(\lambda_j-\lambda)}{x_2(\lambda_j-\lambda)}\frac{1}{y(\lambda-\lambda_l)}. \tag{4.44}$$

For the action of $A_3(\lambda)$ we have a similar expression

$$\begin{aligned}A_3(\lambda)\,\Phi_n(\lambda_1,\ldots,\lambda_n) &= \prod_{k=1}^{n}\frac{x_2(\lambda-\lambda_k)}{x_3(\lambda-\lambda_k)}\Phi_n(\lambda_1,\ldots,\lambda_n)\,A_3(\lambda)\\ &-\varepsilon^n B_3(\lambda)\sum_{j=1}^{n}\frac{1}{y(\lambda-\lambda_j)}\prod_{k=1,k\neq j}^{n}\mathcal{Z}(\lambda_j-\lambda_k)\,\Phi_{n-1}(\lambda_1,\ldots,\hat{\lambda}_j,\ldots,\lambda_n)\,A_2(\lambda_j)\\ &+B_2(\lambda)\sum_{j=2}^{n}\sum_{l=1}^{j-1}H_{jl}(\lambda,\lambda_l,\lambda_j)\prod_{k=1,k\neq j,l}^{n}\mathcal{Z}(\lambda_j-\lambda_k)\,\mathcal{Z}(\lambda_l-\lambda_k)\\ &\times\Phi_{n-2}(\lambda_1,\ldots,\hat{\lambda}_l,\ldots,\hat{\lambda}_j,\ldots,\lambda_n)\,A_2(\lambda_l)\,A_2(\lambda_j)\end{aligned}\tag{4.45}$$

where the scalar functions $H_{jl}(\lambda,\lambda_l,\lambda_j)$ are given by

$$H_{jl}(\lambda,\lambda_l,\lambda_j)=\frac{y_7(\lambda-\lambda_l)}{x_3(\lambda-\lambda_l)}\frac{1}{y(\lambda_l-\lambda_j)}-\frac{y_5(\lambda-\lambda_l)}{x_3(\lambda-\lambda_l)}\frac{1}{y(\lambda-\lambda j)}.\tag{4.46}$$

The action of the operator $A_2(\lambda)$ is more cumbersome

$$\begin{aligned}A_2(\lambda)\,\Phi_n(\lambda_1,\ldots,\lambda_n) &= \varepsilon^n\prod_{k=1}^{n}\frac{z(\lambda-\lambda_k)}{\omega(\lambda-\lambda_k)}\Phi_n(\lambda_1,\ldots,\lambda_n)\,A_2(\lambda)\\ &-\varepsilon^n B_1(\lambda)\sum_{j=1}^{n}\frac{y_5(\lambda-\lambda_j)}{x_2(\lambda-\lambda_j)}\prod_{k=1,k\neq j}^{n}\mathcal{Z}(\lambda_j-\lambda_k)\,\Phi_{n-1}(\lambda_1,\ldots,\hat{\lambda}_j,\ldots,\lambda_n)\,A_2(\lambda_j)\\ &+B_3(\lambda)\sum_{j=1}^{n}\frac{1}{y(\lambda-\lambda_j)}\prod_{k=1,k\neq j}^{n}\mathcal{Z}(\lambda_k-\lambda_j)\,\Phi_{n-1}(\lambda_1,\ldots,\hat{\lambda}_j,\ldots,\lambda_n)\,A_1(\lambda_j)\\ &+\varepsilon^n B_2(\lambda)\Bigg\{\sum_{j=2}^{n}\sum_{l=1}^{j-1}Y_{jl}(\lambda,\lambda_l,\lambda_j)\prod_{k=1,k\neq j,l}^{n}\mathcal{Z}(\lambda_k-\lambda_l)\,\mathcal{Z}(\lambda_j-\lambda_k)\\ &\times\Phi_{n-2}(\lambda_1,\ldots,\hat{\lambda}_l,\ldots,\hat{\lambda}_j,\ldots,\lambda_n)\,A_1(\lambda_l)\,A_2(\lambda_j)\\ &+\sum_{j=2}^{n}\sum_{l=1}^{j-1}F_{jl}(\lambda,\lambda_l,\lambda_j)\prod_{k=1,k\neq j,l}^{n}\mathcal{Z}(\lambda_l-\lambda_k)\,\mathcal{Z}(\lambda_k-\lambda_j)\\ &\times\Phi_{n-2}(\lambda_1,\ldots,\hat{\lambda}_l,\ldots,\hat{\lambda}_j,\ldots,\lambda_n)\,A_1(\lambda_j)\,A_2(\lambda_l)\Bigg\}\end{aligned}\tag{4.47}$$

where we have two more scalar functions

$$\begin{aligned}F_{jl}(\lambda,\lambda_l,\lambda_j) &= \frac{y_5(\lambda-\lambda_l)}{x_2(\lambda-\lambda_l)}\Bigg\{\frac{y_5(\lambda_l-\lambda_j)}{x_2(\lambda_l-\lambda_j)}\frac{1}{y(\lambda-\lambda_l)}+\frac{z(\lambda-\lambda_l)}{\omega(\lambda-\lambda_l)}\frac{1}{y(\lambda-\lambda_j)}\\ &-\frac{y_5(\lambda-\lambda_l)}{x_2(\lambda-\lambda_l)}\frac{1}{y(\lambda_l-\lambda_j)}\Bigg\}\end{aligned}\tag{4.48}$$

$$Y_{jl}(\lambda,\lambda_l,\lambda_j)=\frac{1}{y(\lambda-\lambda_l)}\Bigg\{z(\lambda-\lambda_l)\frac{y_5(\lambda-\lambda_j)}{x_2(\lambda-\lambda_j)}-\frac{y_5(\lambda-\lambda_l)}{x_2(\lambda-\lambda_l)}\frac{y_5(\lambda_l-\lambda_j)}{x_2(\lambda_l-\lambda_j)}\Bigg\}.\tag{4.49}$$

From these relations immediately follows that $\Psi_M(\lambda_1,\ldots,\lambda_M)$ are the eigenstates of $\tau(\lambda)$ with eigenvalues

$$\Lambda_M=x_1(\lambda)^N\prod_{a=1}^{M}z(\lambda_a-\lambda)+\varepsilon^{M+1}x_2(\lambda)^N\prod_{a=1}^{M}\frac{z(\lambda-\lambda_a)}{\omega(\lambda-\lambda_a)}+x_3(\lambda)^N\prod_{a=1}^{M}\frac{x_2(\lambda-\lambda_a)}{x_3(\lambda-\lambda_a)}\tag{4.50}$$

provided their rapidities λ_i, $i = 1, \ldots, M$ satisfy the BA equations

$$\left(\frac{x_1(\lambda_a)}{x_2(\lambda_a)}\right)^N = \varepsilon^{M+1} \prod_{b\neq a=1}^{M} \frac{z(\lambda_a - \lambda_b)}{z(\lambda_b - \lambda_a)} \omega(\lambda_b - \lambda_a) \qquad a = 1, 2, \ldots, M. \tag{4.51}$$

To conclude this section we remark that equations (4.50) and (4.51) reproduce the known results in the literature: using (2.8) we reproduce the BA solution of the IK model [16] and using (2.7) they are the BA solution for the ZF model derived by a fusion procedure in [13–15]. Specifically, in the case of the rational solution and for $\varepsilon = -1$ we obtain previous results derived by the analytical [20] and algebraic [23] BA approach for the rational $osp(1|2)$ vertex model. Furthermore, by using equation (4.7) we recover the expressions derived in the previous section via the coordinate BA.

5. Conclusion

In the first part of this paper we applied the coordinate BA to find the spectra of Hamiltonians associated with three 19-vertex models, including a graded model. This procedure was carried out for periodic boundary conditions.

We believe that the method here presented could also be applied for Hamiltonians associated with higher-states vertex models. For instance, in the quantum spin-chain $s = \frac{3}{2}$ XXZ model we have four states: $|k[\frac{3}{2}]\rangle$, $|k[\frac{1}{2}]\rangle$, $|k[-\frac{1}{2}]\rangle$ and $|k[-\frac{3}{2}]\rangle$. It means that the state $|k(\frac{1}{2})\rangle$ can be parametrized by a plane wave and the states $|k[-\frac{1}{2}]\rangle$ and $|k[-\frac{3}{2}]\rangle$ as two and three states $|k[\frac{1}{2}]\rangle$ at the same site, respectively, multiplied by some weight functions.

These weight functions are responsible for the factorized form of the phase shift of two particles (3.38). In the ZF model we do not have a factored form for the two-pseudoparticle phase shift because its weight function (3.17) is a constant. This means that the state $|k[-]\rangle$ behaves exactly as two states $|k[0]\rangle$ at the same site. This is in agreement with the fact that the ZF model can be constructed by a fusion procedure of the two six-vertex model.

In the second part of this paper we have applied the algebraic BA to find the spectra of the transfer matrices of these three-state vertex models. The method here presented was developed by Tarasov [16] and generalized by Martins [21, 23]. It is general enough to include the ZF model as well as the graded $osp(1|2)$ model.

There are several issues left for future works. A natural extension of this work is to consider these Bethe ansätze with open boundary conditions via reflection matrices. The transfer matrix of the ZF model with the most general diagonal reflection matrix has been diagonalized by Mezincescu *et al* [24] by generalizing the fusion approach used to solve the corresponding model with periodic boundaries. BA equations for both the ZF model and IK model with open boundaries were derived by Yung and Batchelor in [25]. Nevertheless, based on the Tarasov–Martins approach, the algebraic BA of the IK model with a diagonal K-matrix was derived recently by Fan [26].

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